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Spectral properties of random reactance networks and random matrix pencils

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Abstract. Our goal is to study the statistical properies of 'dielectric resonances' which are poles of conductance of a large random LC network. Such poles are a particular example of eigenvalues λ_n of *matrix pencils* $\boldsymbol{H} - \lambda \boldsymbol{W}$, with $\boldsymbol{W}$ being a positive definite matrix and $\boldsymbol{H}$ a random real symmetric one. We first consider spectra of the matrix pencils with independent, identically distributed entries of $\boldsymbol{H}$. Then we concentrate on an infinite-range ('full-connectivity') version of a random LC network. In all cases we calculate the mean eigenvalue density and the two-point correlation function in the framework of Efetov's supersymmetry approach. Fluctuations in spectra turn out to be the same as those provided by the Wigner–Dyson theory of usual random matrices.

1. Introduction

Large random reactance networks (that is networks made of random mixture of capacitances C and inductances L) possess a peculiar property first noticed by Dykhne [1]: they have a finite real conductance and thus can disperse an electric power. The explanation of this somewhat paradoxical property is simple, however. Indeed, if such a network is large enough there always exist circuits of 'resonance type', with (purely imaginary) conductance showing poles at some frequencies. Then the real part of the conductance as a function of frequency consists of a set of δ-like peaks at those resonance frequencies. When the volume of the network grows to infinity, this set becomes increasingly dense. Then, adding an arbitrarily small (infinitesemal, but fixed) active part to all inductances (one may think, e.g. of the inductance L on each bond being in series with a weak resistance R) results in a finite active resistance of the network.

A random mixture of two active conductances is very well studied in correspondence with the bond percolation problem, see e.g. [2]. At the same time, the random reactance networks are relatively less studied.

It is necessary to mention that the random LC (more generally, RL–C) networks emerge in various physical contexts. As was shown long ago by Shender [3], the conductance of the random LC network turns out to be intimately related to properties of collective excitations in spin glasses. The mapping between the two problems is possible by an analogy first noticed by Kirkpatrick [4] between the Kirchhoff law and the equation of motion for the spin operators.

More recently, RL–C networks were claimed to be an adequate model for describing the optical absorption in disordered metal films, see [5] and references therein, which showed some

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unusual features. This fact motivated Luck and collaborators to perform a series of insightful numerical investigations of the two-dimensional RL–C arrays [6, 7].

Since it is the conductance poles (resonances) of the LC networks which dominate the properties of the weakly dissipative RL–C networks, it is natural to try to understand their properties in greater detail. To access those poles it is convenient, following [7, 8], to start with the Kirchhoff equations for the electric potential at vertices i of a network:

$$\sum_j \sigma_{ij}(V_i - V_j) = 0 \tag{1}$$

where the summation goes over all vertices j which are neighbours of a given i. The Kirchhoff equations (1) should be complimented with boundary conditions at the electrodes. For a two-terminal geometry one assumes $V_i = 0$ (resp. $V_i = V$) for the vertices i belonging to the left (resp. right) terminal.

It is useful to introduce the (positive semidefinite) Laplace operator $\boldsymbol{D}$ on the network by

$$(\boldsymbol{D}V)_i = \sum_j (V_j - V_i) \tag{2}$$

with the convention that $V_i = 0$ on both electrodes.

In a random LC network each conductance at frequency $f = \omega/2\pi$ is equal to either $\sigma_0 = \mathrm{i}C\omega$ or $\sigma_1 = (\mathrm{i}L\omega)^{-1}$, with specified probabilities (in what follows we concentrate on the case of equal probability for finding L and C bonds in the network). Then, the Laplace operator can be written as a sum $\boldsymbol{D}_C + \boldsymbol{D}_L$ of its components on the L- and C-bond sets, respectively. It is easy to show that the poles of the conductance occur at frequencies f given by the roots of the equation [7, 8]:

$$\det(\boldsymbol{D}_L - \lambda \boldsymbol{D}) = 0 \tag{3}$$

where we introduce the ratio

$$\lambda = \frac{\sigma_0}{\sigma_0 - \sigma_1} = \frac{(\omega/\omega_0)^2}{1 + (\omega/\omega_0)^2} \qquad 0 \leqslant \lambda \leqslant 1 \tag{4}$$

with $\omega_0 = 1/(LC)^{1/2}$ being a characteristic resonant frequency of the LC network.

The simplest, but very informative way to understand properties of the random binary mixtures, *on average*, is to write an equation for the mean conductance Σ using an effective-medium approximation (EMA). For a network with a coordination number z and equal concentrations of σ_0- and σ_1-bonds it reads as follows [4]:

$$\frac{\Sigma - \sigma_0}{2\sigma_0 + (z-2)\Sigma} + \frac{\Sigma - \sigma_1}{2\sigma_1 + (z-2)\Sigma} = 0. \tag{5}$$

Analysing the solution of this quadratic equation as a function of λ in the interval $0 \leqslant \lambda \leqslant 1$, one finds that generically for $z > 4$ the real part of $\Sigma(\lambda)$ is only non-zero inside some interval $\lambda_{min}(z) \leqslant \lambda \leqslant \lambda_{max}(z)$. According to the discussion above it means that the mean density of resonances is only finite inside that interval.

As useful and simple as it is, the EMA suffers from an essential drawback: it systematically neglects fluctuations, whereas taking the fluctuations into account could be important. For example, as was noted in [3, 7], the existence of sharp edges $\lambda_{min,max}$ was an artifact of the EMA approximation. In fact, the density of the resonances is never exactly zero inside the whole interval $\lambda \in [0, 1]$ due to the so-called Lifshitz tails, which is purely a fluctuation phenomenon. Apart from that, it seems hard to construct the EMA as a systematic approximation with respect to some small parameter, though there are indications that it becomes progressively exact for higher spatial dimensions d [9] and can be an extremely well-working one for many realistic

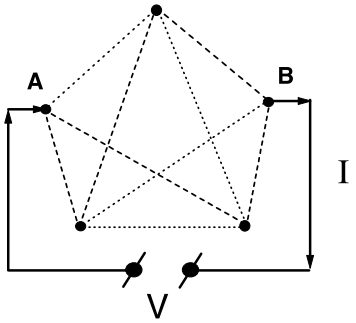

Figure 1. A schematic view of the full-connectivity graph with three internal nodes. One direct bond connecting the terminal nodes A and B is excluded.

systems [4]. Therefore, it is not clear how to correct it systematically or how to take fluctuation effects into account.

These unsatisfactory features should be contrasted with the status of the mean-field approximation in the theory of magnetism, with which EMA shares its main ingredients. As is well known, the mean-field equations become exact for the model with the infinite range of spin interaction. Even for strongly disordered spin systems such as spin glasses the latter model provides an adequate basis for constructing a mean-field theory with many non-trivial properties [10].

It is also natural to try to consider a similar type of model for the disordered reactance networks. This just amounts to considering the Kirchhoff equations (1) on a disordered *full-connectivity* graph of N vertices (nodes) connected by $N(N-1)/2$ edges (bonds), each independently taking a value σ_0 or σ_1.

Actually, we find it convenient to consider a graph with $N+2$ nodes, among them two 'terminal' nodes (labelled A and B, respectively) are singled out by being attached to the external voltage, so that the potential at A is equal to V, whereas the potential at B is kept zero. The rest of the N 'internal' nodes are labelled by indices $i = 1, \ldots, N$ and the corresponding (induced) potentials are denoted with V_i. The nodes are connected in a full-connectivity graph of $(N+2)(N+1)/2 - 1$ bonds, with a single bond being excluded for obvious reasons: that connecting terminals A and B directly, see figure 1.

We then attribute conductances $\sigma_{\mu\nu} = \sigma_{\nu\mu}$, $\mu, \nu = A, B, 1, \ldots, N$ to each bond. It is convenient to introduce three N-component vectors: $\underline{v} = (V_1, \ldots, V_N)^T$, $\underline{\sigma}_A = (\sigma_{A1}, \ldots, \sigma_{AN})^T$, $\underline{\sigma}_B = (\sigma_{B1}, \ldots, \sigma_{BN})^T$ (here we used T to indicate the transposition) and $N \times N$ matrix $\mathbf{\Sigma}$ with the following structure:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{A1} + \sigma_{B1} + \sum_{k\neq 1}^{N} \sigma_{1k} & -\sigma_{12} & \ldots & -\sigma_{1N} \\ -\sigma_{21} & \sigma_{A2} + \sigma_{B2} + \sum_{k\neq 2}^{N} \sigma_{2k} & \ldots & -\sigma_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ -\sigma_{N1} & -\sigma_{N2} & \ldots & \sigma_{AN} + \sigma_{BN} + \sum_{k\neq N} \sigma_{Nk} \end{pmatrix}.$$

In these notations $N+1$ Kirchhoff equations (1) (N for the internal nodes i, and one for the node B) can be written, respectively, as

$$\mathbf{\Sigma}\underline{v} = V\underline{\sigma}_A \qquad \underline{\sigma}_B^T\underline{v} = I \tag{6}$$

where I stands for the total current flowing outwards through the terminal node B. This system of equations can be readily solved yielding the expression for the network conductance:

$$Y_{AB} = \frac{I}{V} = \underline{\sigma}_B^T \mathbf{\Sigma}^{-1} \underline{\sigma}_A. \tag{7}$$

If we now deal with a binary networks when each σ_{ij} is σ_0 with a probability p and σ_1 with the probability $1-p$ it is convenient to introduce the 'symmetric' variables h_{ij} such that

$h_{ij} = -1$ if $\sigma_{ij} = \sigma_0$ and $h_{ij} = 1$ if $\sigma_{ij} = \sigma_1$, so that $\sigma_{ij} = \frac{1}{2}([\sigma_0 + \sigma_1] + [\sigma_1 - \sigma_0]h_{ij})$. In terms of these variables the network conductance equation (7) can be written as

$$y_{ab} = Y_{AB}/\sigma_0 = -\frac{1}{2\lambda}[\underline{h}_B^T - \tilde{\lambda}\underline{e}^T]\frac{1}{\boldsymbol{H} - \tilde{\lambda}\boldsymbol{W}}[\underline{h}_A - \tilde{\lambda}\underline{e}] \tag{8}$$

where $\tilde{\lambda} = 2\lambda - 1$, $\underline{e}^T = (1, 1, \ldots, 1)$, $\underline{h}_B^T = (h_{B1}, \ldots, h_{BN})$ and $N \times N$ matrices $\boldsymbol{W}$, $\boldsymbol{H}$ have the following elements:

$$W_{ij} = (N+2)\delta_{ij} - 1 \qquad H_{ij} = \delta_{ij}\left(h_{Ai} + h_{iB} + \sum_{k\neq i} h_{ik}\right) - (1 - \delta_{ij})h_{ij}. \tag{9}$$

With these expressions in hand we see that resonances of our network are determined by values of $\tilde{\lambda}$ satisfying the condition

$$\det(\boldsymbol{H} - \tilde{\lambda}\boldsymbol{W}) = 0. \tag{10}$$

Of course, it is evident that the full-connectivity construction would be a completely inadequate one for describing the percolation problem which dominates the properties of binary mixtures of two real conductances. We are, however, interested in studying the resonances of disordered LC networks, and the existence of such resonances is in no way precluded by 'all-to-all' geometry. We find that such a model turns out to be, in essence, exactly soluble in the limit $N \gg 1$. We start by deriving the mean density of the resonances which turns out to be a function of the scaled variable $r = \tilde{\lambda}N^{1/2}$. One can already envisage such a scaling from the EMA result, equation (5), which for $z \gg 1$ gives $|\lambda_{min,max} - \frac{1}{2}| \propto \tilde{\lambda}_{min,max} \sim z^{-1/2}$. At the same time, in contrast to EMA, the support of the spectrum in the infinite-range model does not have artificial sharp edges $\tilde{\lambda}_{min,max}$, but rather the mean spectral density $\rho(r)$ smoothly decays to zero as long as $r \to \infty$ as $\rho(r) \propto \mathrm{e}^{-r^2/2}$.

Of more interest is the fact that the infinite-range model opens the possibility to study fluctuations of various quantities. In the present paper we concentrate on spectral fluctuations and, correspondingly, study the two-point correlation function of the resonance densities. The latter turns out to be essentially the same as given by the famous Wigner–Dyson theory of random matrix spectra. This fact favourably agrees with overall numerical results [7] for the resonance spectra of two-dimensional disordered LC networks. The origin of relatively small, but noticable deviations from the Wigner-Dyson statistics detected in [7] remains unclear for us at the moment and could be related to two-dimensional features of the networks studied there which are not captured adequately by our infinite-ranged model. This issue deserves further investigation.

Actually, finding the set of λ satisfing equation (3) or (10) is an example of the generalized eigenvalue problem. The combination of matrices $\boldsymbol{D}_L - \lambda\boldsymbol{D}$ or $\boldsymbol{H} - \tilde{\lambda}\boldsymbol{W}$, in this respect, is known in the mathematical literature as a *pencil of matrices* [11] or just the matrix pencil. The theory of the matrix pencils has many important applications such as, e.g., vibration and bifurcation analysis in complicated structures [12] and game theory [13].

At the same time, it seems that the present knowledge on the statistical properties of generalized eigenvalues of the pencils of random matrices is rather scarce. In [14] the authors consider the mean number and the density of real eigenvalues for a pencil $\boldsymbol{H} - \lambda\boldsymbol{W}$, with both $\boldsymbol{H}$ and $\boldsymbol{W}$ being matrices with all independent real entries and no symmetry conditions imposed. At the same time, our original physical problem has motivated our interest in the pencils formed by real *symmetric* matrices $\boldsymbol{H}, \boldsymbol{W}$, with $\boldsymbol{W}$ being positive definite. It is clear that $\boldsymbol{W} > 0$ ensures all the eigenvalues of the matrix pencils to be real (in the literature such pencils are sometimes called the *regular* ones [11]). Indeed, in that case the generalized eigenproblem: $\boldsymbol{H}\underline{x} = \lambda\boldsymbol{W}\underline{x}$ is equivalent to a usual one: $\boldsymbol{W}^{-1/2}\boldsymbol{H}\boldsymbol{W}^{-1/2}\underline{y} = \lambda\underline{y}$, $\underline{y} = \boldsymbol{W}^{1/2}\underline{x}$, with $\tilde{\boldsymbol{H}} = \boldsymbol{W}^{-1/2}\boldsymbol{H}\boldsymbol{W}^{-1/2}$ being a real symmetric matrix.

The mean eigenvalue density for matrices of similar types (when both $\boldsymbol{W}$ and $\boldsymbol{H}$ are random) was studied in some generality starting from the work by Marchenko and Pastur, see [15]. In fact, the mean eigenvalue density can be found for the ensemble $\tilde{\boldsymbol{H}}$, with $\boldsymbol{H}$ being a real symmetric matrix with independent, identically distributed (IID) entries using a generalized version of the results of Pastur [16]. We are, however, not aware of any systematic study of spectral correlations of regular pencils of the random matrices.

This is in contrast to the very intensive research on eigenvalues of the random matrices (which is a particular case of $\boldsymbol{W} \equiv \mathbf{1}$) performed in recent years in the domain of theoretical and mathematical physics [17]. Below, we summarize the most important facts found from these studies.

As is well established [18–20, 22, 23], the statistical properties of real eigenvalues X_i of large $N \times N$ self-adjoint random matrices $\boldsymbol{H}$ are to a large extent *universal*, i.e. independent of the details of the distributions $\mathcal{P}(\boldsymbol{H})$ of their entries.

It is important to mention the existence, in general, of two different characteristic scales in the random matrix spectra: the *global* one and the *local* one. The global scale is that on which the eigenvalue density, defined as $\rho(X) = \frac{1}{N} \operatorname{Tr} \delta(X - \boldsymbol{H})$, changes appreciably with its argument X when averaged over $\mathcal{P}(\boldsymbol{H})$. For matrices whose spectrum has a finite support in the interval $X \in [A, B]$ the global scale is just the length of this interval.

In contrast, the local scale is that determined by the typical separation $\Delta = \langle X_i - X_{i-1} \rangle$ between neighbouring eigenvalues situated around a point X, with the brackets standing for the statistical averaging. It is therefore given by $\Delta = (\langle N\rho(X) \rangle)^{-1}$. If we are interested in those values of X that are sufficiently far from the edges of the spectra the global scale is, roughly speaking, a factor of N larger than the local one. In other words, the mean density $\langle \rho(X) \rangle$ can be considered as a constant one on the scale Δ.

The degree of universality is essentially dependent on the chosen scale.

As to the global scale universality, first of all one can mention that for the matrices with IID entries under quite general conditions (see, e.g., [24] and references therein) the mean density is given in the limit $N \to \infty$ by the so-called 'Wigner semicircle law':

$$\langle \rho(X) \rangle = \frac{1}{2\pi a^2} \sqrt{4a^2 - X^2} = \frac{1}{N\Delta}. \tag{11}$$

In this expression the parameter a just sets the global scale in the sense defined above. It is determined by the expectation value $a^2 = \langle \frac{1}{N} \operatorname{Tr} \boldsymbol{H}^2 \rangle$. It is generally accepted to scale the entries $H_{ij} \sim 1/N^{1/2}$, i.e. in such a way that a stays finite when $N \to \infty$, the local spacing between eigenvalues in the neighbourhood of the point X therefore being $\Delta \propto 1/N$.

From the point of view of universality the semicircular eigenvalue density is not extremely robust. One can most easily violate it by considering an important class of the so-called 'invariant ensembles' characterized by a probability density of the form $\mathcal{P}(\boldsymbol{H}) \propto \exp -N \operatorname{Tr} V(\boldsymbol{H})$, with $V(\boldsymbol{H})$ being an even polynomial. The corresponding eigenvalue density turns out to be highly nonuniversal and is determined by the particular form of the potential $V(H)$. Only for $V(\boldsymbol{H}) = \boldsymbol{H}^2$ is it given by the semicircular law, equation (11). Actually, any 'deformation' $\boldsymbol{H}_1 = \boldsymbol{H}_0 + \boldsymbol{H}$ of the ensemble $\boldsymbol{H}$ with IID entries by a given *fixed* matrix $\boldsymbol{H}_0$ results in the mean eigenvalue density belonging to a family of 'deformed semicircular laws' as discovered by Pastur [16].

Moreover, one can easily have a non-semicircular eigenvalue density even for real symmetric matrices $\boldsymbol{S}$; $S_{ij} = S_{ji}$ with IID entries, if one keeps the mean number of non-zero entries, p, per column to be of the order of unity when performing the limit $N \to \infty$. This is a characteristic feature of the so-called *sparse* random matrices [22,25,26] characterized

by the following probability density of a given entry S_{ij}:

$$\mathcal{P}(S_{ij}) = \left(1 - \frac{p}{N})\delta(S_{ij}\right) + \frac{p}{N}h(S_{ij}) \tag{12}$$

where $h(s) = h(-s)$ is an arbitrary even distribution function satisfying the conditions: $h(0) < \infty$; $\int h(s)s^2\,\mathrm{d}s < \infty$.

Remarkably, a much more profound universality emerges for the two-point correlation spectral function defined as

$$\langle\rho(X_1)\rho(X_2)\rangle_c = \langle\rho(X_1)\rangle\delta(X_1 - X_2) - \mathcal{Y}_2(X_1, X_2) \tag{13}$$

where we defined the connected part of the correlation function in a usual manner: $\langle AB\rangle_c = \langle AB\rangle - \langle A\rangle\langle B\rangle$. The nontrivial part of the spectral correlator is called the *cluster function* $\mathcal{Y}_2(X_1, X_2)$. It is one of the most informative statistical measures of the spectra [28]. It turns out that the global scale behaviour of $\mathcal{Y}_2(X_1, X_2)$ (i.e. one for the distance S being comparable with the support of the spectrum in the limit $N \to \infty$) is already rather universal. It is the same for all the 'invariant ensembles' [19] and for those with IID matrix elements [23] and is determined only by the positions of the edge points $[A, B]$ of the spectrum†.

Even more interesting is the fact that universality of the correlation function equation (13) (as well as of all higher correlation functions) extends to the local scale, i.e. for the distances $|X_1 - X_2|$ comparable with Δ. This fact was rigorously proved for the unitary invariant ensembles [18], extended to the unitary 'deformed' ensembles [21] and heuristically verified for other invariant ensembles [20] as well as for the ensembles of sparse matrices [22]‡. The particular form of the cluster function is different from that typical for the global scale. In general, it is dictated by *global symmetries* of the random matrices, e.g. if they are complex Hermitian or real symmetric [28]. All specific (nonuniversal) properties are encoded in the value of local spacing Δ. For the distances S such that $S \gg \Delta$ local expressions match with the global one, the latter taken at distances $S \ll [A - B]$.

It turns out that it is the *local scale* universality that is most relevant for real physical systems [30]. Namely, the statistics of highly excited bound states of *closed* quantum chaotic systems of quite different microscopic nature turn out to be independent of the microscopic details when sampled on the energy intervals which are large in comparison with the mean level separation, but smaller than the energy scale related by the Heisenberg uncertainty principle to the relaxation time necessary for a classically chaotic system to reach equilibrium in phase space [31]. Moreover, the spectral correlation functions turn out to be exactly those which are provided by the theory of large random matrices on the *local* scale [32, 33], with different symmetry classes corresponding to presence or absence of the time-reversal symmetry.

Motivated by a lack of general theoretical results for spectral properties of the pencils of random matrices, in section 2 we start by considering an abstract pencil, with real symmetric $\boldsymbol{H}$ belonging to the Gaussian ensemble of random matrices with IID entries and W being a real symmetric positive definite one with fixed given entries. We derive an expression for the mean spectral density of such a pencil and demonstrate that the correlation properties on the local scale (in the sense defined above) are the same as given by the Wigner–Dyson expressions. Then, in section 3 we extend our consideration to the case of resonance statistics in the random

† These so-called 'wide universal correlations' are, however, quite sensitive to the number of the intervals supporting nonzero mean eigenvalue density, for an example, see [27].

‡ Strictly speaking, the form of the correlation function of eigenvalue densities for sparse matrices was shown to be identical to that known for the corresponding Gaussian ensemble provided p exceeds some critical value $p = p_l$. The 'threshold' value p_l is nonuniversal and depends on the form of the distribution $\mathcal{P}(\dot{H})$ [22]. However, direct numerical simulations, see [29], show that the actual value is $1 < p_l < 2$. Thus, even the existence of two nonvanishing elements per row already ensure that the corresponding statistics belongs to the Gaussian universality class.

infinite-range LC network, by deriving the mean density and the two-point spectral correlation function for the resonances.

2. Wigner–Dyson universality for spectra of regular matrix pencils

To study spectral properties of a regular matrix pencil $\boldsymbol{H} - \lambda \boldsymbol{W}$ we employ the Efetov supersymmetry approach [34, 35]. A pedestrian introduction to the method can be found in [36].

A convenient starting point is a representation of the spectral density $\rho(\lambda)$ of real eigenvalues λ_n for the equivalent symmetric eigenvalue problem for the matrix $\tilde{\boldsymbol{H}} = \boldsymbol{W}^{-1/2}\boldsymbol{H}\boldsymbol{W}^{-1/2}$ in the following form:

$$\rho(\lambda) = \frac{1}{N}\sum_{n=1}^{N}\delta(\lambda - \lambda_n) = \mp \lim_{\epsilon\to+0}\frac{1}{\pi N}\mathrm{Im}\ \mathrm{Tr}\,\frac{1}{(\lambda \pm \mathrm{i}\epsilon)\mathbf{1} - \tilde{\boldsymbol{H}}} \tag{14}$$

which after a trivial manipulation can be rewritten as a derivative

$$\rho(\lambda) = \mp\mathrm{Im}\,\frac{1}{\pi N}\lim_{\epsilon\to+0}\lim_{J\to 0}\frac{\partial}{\partial J}\mathcal{Z}_{\pm}(J,\lambda) \tag{15}$$

of a generation function $\mathcal{Z}_{\pm}(J,\lambda)$ defined as

$$\mathcal{Z}_{\pm}(J,\lambda) = \frac{\det[(\lambda \pm \mathrm{i}\epsilon + J)\boldsymbol{W} - \boldsymbol{H}]}{\det[(\lambda \pm \mathrm{i}\epsilon)\boldsymbol{W} - \boldsymbol{H}]}. \tag{16}$$

In a completely analogous way one expresses the two-point spectral correlation function ('cluster function', see equation (13))

$$-\mathcal{Y}_2(\lambda_1,\lambda_2) = \langle\rho(\lambda_1)\rho(\lambda_2)\rangle - \langle\rho(\lambda_1)\rangle\langle\rho(\lambda_2)\rangle = \frac{1}{2\pi^2}\mathrm{Re}\,\mathcal{K}_{con}(\lambda_1,\lambda_2) \tag{17}$$

as

$$\mathcal{K}_{con}(\lambda_1,\lambda_2) = \frac{1}{N^2}\lim_{\epsilon\to+0}\lim_{J_{1,2}\to 0}\frac{\partial^2}{\partial J_1\partial J_2}[\langle\mathcal{Z}_{-}(J_1,\lambda_1)\mathcal{Z}_{+}(J_2,\lambda_2)\rangle - \langle\mathcal{Z}_{-}(J_1,\lambda_1)\rangle\langle\mathcal{Z}_{+}(J_2,\lambda_2)\rangle] \tag{18}$$

where the angular brackets stand for the ensemble averaging and we assume that $\lambda_1 \neq \lambda_2$. The expressions above are actually valid in the limit $N \to \infty$, where one can show that

$$\lim_{\epsilon\to+0}\lim_{J_{1,2}\to 0}\frac{\partial^2}{\partial J_1\partial J_2}[\langle\mathcal{Z}_{\pm}(J_1,\lambda_1)\mathcal{Z}_{\pm}(J_2,\lambda_2)\rangle - \langle\mathcal{Z}_{\pm}(J_1,\lambda_1)\rangle\langle\mathcal{Z}_{\pm}(J_2,\lambda_2)\rangle] = 0$$

and we used this fact in equation (17).

To facilitate the ensemble averaging we represent the ratio of the two determinants in equation (16) as the Gaussian integral

$$\mathcal{Z}_{\pm}(J,\lambda) = (-1)^N\int\prod_{i=1}^{N}[\mathrm{d}\Psi_i(\pm)]\exp\left\{\pm\frac{\mathrm{i}}{2}\sum_{i,j}^{N}\Psi_i^{\dagger}(\pm)[W_{ij}(\lambda \pm \mathrm{i}\epsilon + J\hat{k}) - H_{ij}]\Psi_j(\pm)\right\} \tag{19}$$

over four-component supervectors $\Psi_i(\pm)$,

$$\begin{aligned}
\Psi_i(\pm) &= \begin{pmatrix}\vec{R}_i(\pm)\\ \vec{\eta}_i(\pm)\end{pmatrix} \qquad & \vec{R}_i(\pm) &= \begin{pmatrix} r_i(\pm)\\ r_i^*(\pm)\end{pmatrix}\\
\vec{\eta}_i(\pm) &= \begin{pmatrix}\chi_i(\pm)\\ \chi_i^*(\pm)\end{pmatrix} \qquad & \mathrm{d}\Psi_i &= \frac{\mathrm{d}r_i\mathrm{d}r_i^*}{2\pi}\mathrm{d}\chi_i^*\,\mathrm{d}\chi_i
\end{aligned} \tag{20}$$

with the components $r_i(+)$, $r_i(-)$; $i = 1, 2, \ldots, N$ being complex commuting variables and $\chi_i(+)$, $\chi_i(-)$ forming the corresponding Grassmannian parts of the supervectors $\Psi_i(\pm)$. A 4×4 diagonal supermatrix $\hat{k} = \mathrm{diag}(0, 0, 1, 1)$ takes care of the absence of the source J in the denomenator of the generating function, equation (16).

Since we are finally dealing with the averaged product of two such generating functions it is convenient to introduce the 8-component supervectors

$$\Phi_i = \begin{pmatrix} \Psi_i(+) \\ \Psi_i(-) \end{pmatrix} \tag{21}$$

and finally $\Phi = (\Phi_1, \ldots, \Phi_N)^T$, as well as the supermatrices $\hat{\Lambda} = \mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$ and $\hat{J}_k = \mathrm{diag}(J_1\hat{k}, J_2\hat{k})$.

It is also useful to remember that we expect to have non-trivial spectral correlations on the scale comparable with the mean spacing between neigbouring eigenvalues, i.e. when $\lambda_1 - \lambda_2 \propto 1/N$. Correspondingly, we introduce $\lambda_1 = \lambda - s/2N$, $\lambda_2 = \lambda + s/2N$ and henceforth consider s to be of the order of unity. All this allows us to write the product of two generating functions as

$$\mathcal{Z}^{\lambda,s}(J_1, J_2) = \mathcal{Z}_-(J_1, \lambda_1)\mathcal{Z}_+(J_2, \lambda_2) \int \mathrm{d}\Phi \times \exp\left\{ -\frac{\mathrm{i}}{2}\Phi^\dagger \hat{\Lambda} \otimes \boldsymbol{H}\Phi + \frac{\mathrm{i}}{2}\Phi^\dagger[\lambda\hat{\Lambda} - (\epsilon - \mathrm{i}s/2N)\hat{1} + \hat{J}_k] \otimes \boldsymbol{W}\Phi \right\}. \tag{22}$$

This expression can be easily averaged over the Gaussian distribution of $\boldsymbol{H}$ by the chain of identities

$$\langle \mathrm{e}^{-\frac{\mathrm{i}}{2}\Phi^\dagger \hat{\Lambda} \otimes \boldsymbol{H}\Phi} \rangle = \mathrm{e}^{-\frac{a^2}{4N}\,\mathrm{Str}\,\hat{A}^2} = \int \mathrm{d}\hat{Q} \exp\left[-\frac{N}{4}\,\mathrm{Str}\,\hat{Q}^2 - \mathrm{i}\frac{a}{2}\Phi^\dagger \hat{\Lambda}\hat{Q}\Phi \right] \tag{23}$$

where a determines the variance of matrix elements of $\boldsymbol{H}$, see equation (11) and $\hat{A} = \hat{\Lambda}^{1/2}(\sum_i \Phi_i \otimes \Phi^\dagger)\hat{\Lambda}^{1/2}$. The last relation which trades the term in the exponent quartic with respect to Φ for an auxilliary integration over the set of supermatrices $\hat{Q}$ is known as the Hubbard–Stratonovich transformation and plays a cornerstone role in the whole method. After substituting equation (23) back into averaged equation (22) and changing the order of integrations one performs the (Gaussian) integral over Φ explicitly. It turns out that in order to justify all these operations formally, one has to restrict the supermatrices $\hat{Q}$ to a manifold paramatrized as $\hat{Q} = \hat{T}^{-1}\hat{P}\hat{T}$, with $\hat{P}$ being a block-diagonal Hermitian supermatrix and $\hat{T}$ belonging to a certain graded coset space $UOSP(2, 2/4)/UOSP(2/2) \otimes UOSP(2/2)$. A detailed discussion of this fact and an explicit parametrization of the $\hat{T}$ matrices can be found in [34–37].

The resulting expression for the averaged product of the generation functions turns out to be dependent only on eigenvalues w_i, $i = 1, \ldots, N$ of the matrix $\boldsymbol{W}$ (this fact can be traced back to the 'rotational invariance' of the Gaussian orthogonal ensemble (GOE) formed by $\boldsymbol{H}$) and has the following form:

$$\langle \mathcal{Z}^{\lambda,s}(J_1, J_2) \rangle = \int \mathrm{d}\hat{Q} \exp\left[-\frac{N}{4}\,\mathrm{Str}\,\hat{Q}^2 - \frac{1}{2}\sum_{i=1}^{N} \mathrm{Str}\ln\{(\lambda\hat{1} - v_i\hat{Q}) + (\mathrm{i}\epsilon + s/2N + \hat{J}_k)\hat{\Lambda}\} \right] \tag{24}$$

where $v_i = a/w_i$.

Since we are interested in the limit $(\epsilon, J, 1/N) \to 0$, we can expand the logarithm in the exponent correspondingly:

$$\mathrm{Str}\ln\{(\lambda\hat{1} - v_i\hat{Q}) + (\mathrm{i}\epsilon + s/2N + \hat{J}_k)\hat{\Lambda}\}$$

$$= \operatorname{Str}\ln(\lambda\hat{1} - v_i\hat{Q}) + \operatorname{Str}(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{\Lambda}(\mathrm{i}\epsilon + s/2N + \hat{J}_k)$$
$$-\tfrac{1}{2}\operatorname{Str}[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{\Lambda}\hat{J}_k]^2 + \cdots \tag{25}$$

so that differentiating the exponent over the sources $J_{1,2}$ yields the pre-exponential factors:

$$\lim_{J_{1,2}\to 0}\frac{\partial}{\partial J_1}\exp\{\ldots\} \propto \sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}+\hat{1}}{2}\right] \tag{26}$$

and

$$\lim_{J_{1,2}\to 0}\frac{\partial^2}{\partial J_1\partial J_2}\exp\{\ldots\} \propto 2\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}+\hat{1}}{2}(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}-\hat{1}}{2}\right]$$
$$+\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}+\hat{1}}{2}\right]\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}-\hat{1}}{2}\right]. \tag{27}$$

In the limit $N \gg 1$, each sum over i is of the order of N, so that a contribution of second term in the expression above is larger by a factor of N than the contribution of the first one. In other words, we could restrict ourselves by terms linear in sources $J_{1,2}$ in expansion (25) above. We will make use of this fact in the next section when considering a more complicated situation.

Taking all this into account, we arrive at the following integral representation:

$$\mathcal{K}(\lambda_1,\lambda_2) = \frac{1}{N^2}\lim_{\epsilon\to+0}\lim_{J_{1,2}\to 0}\frac{\partial^2}{\partial J_1\partial J_2}\langle\mathcal{Z}_-(J_1,\lambda_1)\mathcal{Z}_+(J_2,\lambda_2)\rangle$$
$$= \frac{1}{4N^2}\lim\epsilon\to 0\int \mathrm{d}Q$$
$$\times\exp\left[-N\mathcal{L}(\hat{Q}) - \frac{\mathrm{i}}{2}(\epsilon - \mathrm{i}s/2N)\sum_i\operatorname{Str}(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{\Lambda}\right]$$
$$\times\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}+\hat{1}}{2}\right]\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}-\hat{1}}{2}\right] \tag{28}$$

where

$$\mathcal{L}(\hat{Q}) = \tfrac{1}{2}\operatorname{Str}\hat{Q}^2 + \frac{1}{N}\sum_i\operatorname{Str}\ln(\lambda\hat{1} - v_i\hat{Q}). \tag{29}$$

In the same way one obtains the expression for the mean spectral density:

$$\rho(\lambda) = \frac{1}{N\pi}\lim_{\epsilon\to 0}\operatorname{Im}\int \mathrm{d}Q\sum_{i=1}^{N}\operatorname{Str}\left[(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{k}\frac{\hat{\Lambda}+\hat{1}}{2}\right]$$
$$\times\exp\left[-N\mathcal{L}(\hat{Q}) - \frac{\mathrm{i}}{2}\epsilon\sum_i\operatorname{Str}(\lambda\hat{1} - v_i\hat{Q})^{-1}\hat{\Lambda}\right]. \tag{30}$$

In the limit $N\to\infty$ the integrals over $\hat{Q}$ in the expressions above are dominated by saddle points of the 'action' $\mathcal{L}(\hat{Q})$ equation (29) which satisfy the equation

$$\hat{Q} = \frac{1}{N}\sum_{i=1}^{N}\frac{v_i}{\lambda\hat{1} - v_i\hat{Q}} \tag{31}$$

relevant solutions of which belonging to the integration domains $\hat{Q} = \hat{T}^{-1}\hat{P}\hat{T}$ are parametrized as follows:

$$\hat{Q}_{s.p.} = t\hat{1} + \mathrm{i}q\hat{T}^{-1}\hat{\Lambda}\hat{T} \tag{32}$$

where the real parameters t, q satisfy for $q \neq 0$ the following system of two equations:

$$t = \frac{1}{N}\sum_i \frac{v_i(\lambda - v_i t)}{(\lambda - v_i t)^2 + q^2 v_i^2} \qquad \text{and} \qquad 1 = \frac{1}{N}\sum_i \frac{v_i^2}{(\lambda - v_i t)^2 + q^2 v_i^2}. \tag{33}$$

Let us note, that the saddle-point solutions, equation (32), for $q \neq 0$ form a continuous manifold parametrized by the supermatrices $\hat{T}$.

Using these expressions it is easy to invert the matrix $(\lambda\hat{1} - v_i\hat{Q}_{s.p})$ and check that

$$\sum_{i=1}^{N} \mathrm{Str}\left[(\lambda\hat{1} - v_i\hat{Q}_{s.p})^{-1}\hat{k}\frac{\hat{\Lambda} \pm \hat{1}}{2}\right] = \pm 2\frac{\lambda - v_i t}{(\lambda - v_i t)^2 + q^2 v_i^2} + \mathrm{i}q\frac{v_i}{(\lambda - v_i t)^2 + q^2 v_i^2}\,\mathrm{Str}\left[\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{k}\frac{\hat{1} \pm \hat{\Lambda}}{2}\right] \tag{34}$$

$$\sum_i \mathrm{Str}(\lambda\hat{1} - v_i\hat{Q}_{s.p})^{-1}\hat{\Lambda} = \mathrm{i}\,\mathrm{Str}[\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}]\sum_i \frac{v_i q}{(\lambda - v_i t)^2 + q^2 v_i^2}. \tag{35}$$

So the problem amounts to substituting these expressions into the integrand of equations (28), (30) and performing the remaining integration over the coset space parametrized by the matrices $\hat{T}$. The method is described in greater detail in [34–37] and we refer the interested reader to those papers.

Here, we mention only a few of the most important aspects. First of all, the integral in equation (30) is given by the so-called PSEW (Parisi–Sourlas–Efetov–Wegner) theorem due to a specific symmetry of the integrand (in the limit $\epsilon \to 0$ the integrand contains only a part of the integration variables due to the projector $(\hat{\Lambda} + 1)/2$). The resulting mean eigenvalue density is merely given by

$$\rho(\lambda) = \frac{q}{\pi N}\sum_i \frac{v_i}{(\lambda - v_i t)^2 + q^2 v_i^2}. \tag{36}$$

In fact, the system of equations (33), (36) turns out to be equivalent to a particular case of some general results obtained by Pastur and Girko [16].

Substituting expression (36) for the mean spectral density into equations (34) and the resulting formulae, further to the integral equations (28), (29) for the averaged generating function, one can express the 'connected' part of the two-point correlation function $\mathcal{K}_{con}(\lambda_1, \lambda_2)$ as

$$\mathcal{K}_{con}(\lambda_1, \lambda_2) = \frac{\pi^2\rho^2(\lambda)}{4}\int \mathrm{d}\mu\,(T)\,\mathrm{Str}\left[\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{k}\frac{\hat{1} + \hat{\Lambda}}{2}\right]\mathrm{Str}\left[\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{k}\frac{\hat{1} - \hat{\Lambda}}{2}\right] \times \exp\left\{-\frac{\mathrm{i}\pi\rho(\lambda)}{4}s\,\mathrm{Str}[\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}]\right\} \tag{37}$$

with $\mathrm{d}\mu\,(T)$ being an appropriate invariant measure on the coset space. When deriving that expression from equations (28), (29), (34) we noticed that the 'disconnected' part of the correlation function is again given by PSEW theorem and exactly cancels the contribution from the few terms in the integrand which are not proportional to the mean density $\rho(\lambda)$.

Expression (37) is our main result and is quite remarkable: its form coincides exactly with the corresponding expression for the two-point correlation function of random matrices from GOE, see e.g. [37]. This fact was first demonstrated by Efetov who managed to perform a non-trivial integration over the coset-space explicitly [34] and found that it reproduced the

famous Dyson expression† for the two-point function [28]:

$$\mathcal{Y}_2(\lambda_1,\lambda_2)=\left(\frac{\sin s_e}{s_e}\right)^2+\frac{\mathrm{d}}{\mathrm{d}s_e}\left(\frac{\sin s_e}{s_e}\right)\int_1^\infty \mathrm{d}t\,\frac{\sin(s_e t)}{t} \tag{38}$$

where $s_e=\pi\rho(\lambda)s$ is a spectral distance s/N measured in units of the local mean spacing $\Delta=1/[N\rho(\lambda)]$.

To end this section we just mention that our method of deriving equation (37) can be easily generalized to matrix pencils whose random part $\boldsymbol{H}$ is an arbitrary matrix (real symmetric or Hermitian) with IID entries following the lines of [22, 26]. In the same way one can show that all higher correlation functions also coincide with GOE expressions.

3. Spectral properties of a 'full connectivity' LC network

Now we are well prepared to calculate the mean density of resonances and the corresponding two-point spectral correlation function of the 'full-connectivity' LC network as defined by equations (9), (10). Actually, it is convenient to rescale both matrices in equation (9) as: $\boldsymbol{H}\to\frac{1}{N^{1/2}}\boldsymbol{H}$, $\boldsymbol{W}\to\frac{1}{N^{1/2}}\boldsymbol{W}$. Obviously, this transformation does not change generalized eigenvalues of the pencil $\boldsymbol{H}-\tilde{\lambda}\boldsymbol{W}$ but facilitates bookkeeping of the leading terms in various expansions.

In our consideration we follow the pattern of the previous section and introduce the generation function identical to equation (22):

$$\begin{aligned}\mathcal{Z}^{r,s}(J_1,J_2)=&\int\prod_i \mathrm{d}\Phi_i\\ &\times\exp\left\{-\frac{\mathrm{i}}{2N^{1/2}}\sum_{i=1}^{N}\Phi_i^\dagger\left[(h_{Ai}+h_{Bi})\hat{\Lambda}-2\mathrm{i}\tilde{\epsilon}\hat{1}-2\left(\frac{r}{N^{1/2}}\hat{\Lambda}+\hat{J}\right)\right]\Phi_i\right\}\\ &\times\exp\left\{\frac{\mathrm{i}}{2N^{1/2}}\sum_{i<j}^{N}(\Phi_i^\dagger-\Phi_j^\dagger)\left[\frac{r}{N^{1/2}}\hat{\Lambda}+\hat{J}_k+\mathrm{i}\tilde{\epsilon}\hat{1}+h_{ij}\hat{\Lambda}\right](\Phi_i-\Phi_j)\right\}\end{aligned} \tag{39}$$

where we used that

$$\tfrac{1}{2}\sum_{i<j}h_{ij}(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)=-\sum_{i<j}h_{ij}\Phi_i^\dagger\hat{B}\Phi_j+\tfrac{1}{2}\sum_i\Phi_i^\dagger\hat{B}\Phi_i\left(\sum_{k\neq i}h_{ik}\right)$$

for any supermatrix $\hat{B}$ as long as $\Phi_i^\dagger\hat{B}\Phi_j=\Phi_j^\dagger\hat{B}\Phi_i$. We also envisaged that for the full-connectivity network the resonance density is nonvanishing as long as $\tilde{\lambda}\sim 1/N^{1/2}$ and introduced the scaled variable $r=\tilde{\lambda}N^{1/2}$ (see the introduction). Then, the typical spacing between neighbouring resonances should be of the order of $\Delta\sim N^{-3/2}$. Since we can expect a nontrivial spectral correlation on the distances $S=\tilde{\lambda}_2-\tilde{\lambda}_1$ comparable with Δ we introduced the quantity $\tilde{\epsilon}=\epsilon-\mathrm{i}s/(2N^{3/2})$ and consider s to be of the order of unity. All other notations coincide with those in equation (22).

Now we have to perform the averaging over $h_{\mu\nu}$, $(\mu,\nu)=A,B,1,\ldots,N$, each taking values of ±1 with the equal probabilities. This is done in the limit $N\to\infty$ as follows:

$$\begin{aligned}&\prod_{i<j}\left\langle\exp\left\{\frac{\mathrm{i}}{2N^{1/2}}h_{ij}(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)\right\}\right\rangle\\ &\qquad=\prod_{i<j}\frac{1}{2}\left[\exp\left\{\frac{\mathrm{i}}{2N^{1/2}}(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)\right\}\right.\end{aligned}$$

† The corresponding (unnumbered) expression in the first of the papers in [22] (after equation (45)) contains a misprint.

$$+\exp\left\{-\frac{\mathrm{i}}{2N^{1/2}}(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)\right\}\Bigg]$$
$$=\prod_{i<j}\left\{1-\frac{1}{8N}[(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)]^2+\cdots\right\}$$
$$\approx\exp\left\{-\frac{1}{16N}\sum_{i,j}[(\Phi_i^\dagger-\Phi_j^\dagger)\hat{B}(\Phi_i-\Phi_j)]^2\right\} \tag{40}$$

and in similar way we can also average over $h_{A,i}$, $h_{B,i}$. Now introducing the notations

$$\mathcal{K}(\Phi_a,\Phi_b)=-4ir(\Phi_a^\dagger-\Phi_b^\dagger)\hat{\Lambda}(\Phi_a-\Phi_b)+[(\Phi_a^\dagger-\Phi_b^\dagger)\hat{\Lambda}(\Phi_a-\Phi_b)]^2 \tag{41}$$
$$f(\Phi)=\frac{i}{N^{1/2}}\Phi^\dagger\left(\mathrm{i}\tilde{\epsilon}\hat{1}+\frac{r}{N^{1/2}}\hat{\Lambda}+\hat{J}_k\right)\Phi-\frac{1}{4N}(\Phi^\dagger\hat{\Lambda}\Phi)^2 \tag{42}$$

we can rewrite the averaged generating function as

$$\mathcal{Z}(J_1,J_2)_{av}=\int\prod_i \mathrm{d}\Phi_i\exp\left\{\sum_i f(\Phi_i)-\frac{1}{16N}\sum_{ij}\mathcal{K}(\Phi_i,\Phi_j)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2N^{1/2}}\sum_{i<j}(\Phi_i^\dagger-\Phi_j^\dagger)[\mathrm{i}\tilde{\epsilon}\hat{1}+\hat{J}_k](\Phi_i-\Phi_j)\right\}. \tag{43}$$

For the case of Gaussian real symmetric matrices with independent entries, $\boldsymbol{H}$, considered in the previous section, the key point allowing us to make progress was the exploitation of the Hubbard–Stratonovich decoupling; see equation (23). In a similar way, we proceed here using a *functional generalization* of the Hubbard-Stratonovich identity suggested by us earlier [22] in the context of studies of the sparse random matrices:

$$\exp\left\{-\frac{1}{16N}\sum_{i,j}\mathcal{K}(\Phi_i,\Phi_j)\right\}=\int\mathcal{D}(g)$$
$$\times\exp\left\{-\frac{N}{16}\int\mathrm{d}\Phi_a\,\mathrm{d}\Phi_b\,g(\Phi_a)\mathcal{C}(\Phi_a,\Phi_b)g(\Phi_b)+\frac{\mathrm{i}}{8}\sum_{i=1}^{N}g(\Phi_i)\right\} \tag{44}$$

where the kernel $\mathcal{C}(\Phi_a,\Phi_b)$ is, in a sense, the inverse of a (symmetric) kernel $\mathcal{K}(\Phi_a,\Phi_b)$:

$$\int\mathrm{d}\Psi\,\mathcal{K}(\Phi_a,\Psi)\mathcal{C}(\Psi,\Phi_b)=\delta(\Phi_a,\Phi_b) \tag{45}$$

and $\delta(\Phi_a,\Phi_b)$ plays the role of a δ-functional kernel in a space spanned by the functions $g(\Phi)$. Some hints towards the understanding of the identities (44), (45) are given in the appendix.

With the help of these relations one easily brings the averaged generation function equation (43) to the form

$$\langle\mathcal{Z}^{r,s}(J_1,J_2)\rangle=\int\mathcal{D}(g)\exp\{-N\mathcal{L}(g)+\delta\mathcal{L}_1(g)\} \tag{46}$$

$$\mathcal{L}(g)=\tfrac{1}{16}\int\mathrm{d}\Phi_a\,\mathrm{d}\Phi_b\,g(\Phi_a)\mathcal{C}(\Phi_a,\Phi_b)g(\Phi_b)-\ln\int\mathrm{d}\Phi\,\mathrm{e}^{\frac{\mathrm{i}}{8}g(\Phi)+f(\Phi)} \tag{47}$$

$$\delta\mathcal{L}_1(g)=\ln\Bigg[\int\prod_i\mathrm{d}\Phi_i\exp\left\{\sum_i\left(\frac{\mathrm{i}}{8}g(\Phi_i)+f(\Phi_i)\right)\right.$$
$$\left.+\frac{\mathrm{i}}{2N^{1/2}}\sum_{i<j}(\Phi_i^\dagger-\Phi_j^\dagger)[\mathrm{i}\tilde{\epsilon}\hat{1}+\hat{J}_k](\Phi_i-\Phi_j)\right\}\Bigg]$$
$$\times\Bigg[\int\prod_i\mathrm{d}\Phi_i\exp\sum_i\left(\frac{\mathrm{i}}{8}g(\Phi_i)+f(\Phi_i)\right)\Bigg]^{-1}. \tag{48}$$

As is clear from these expressions, the term $\mathcal{L}_1(g)$ is only a small correction to the main term in the exponent of the functional integral when $N \to \infty$, $J, \epsilon \to 0$. Therefore, the functional integration over g can be performed by the saddle-point method (cf equations (28), (30)). The saddle-point configuration $g_s(\Phi)$ can be found by requiring the vanishing variation of the 'action' $\mathcal{L}(g)$ and satisfies the following equation:

$$g(\Phi_a) = \mathrm{i}\frac{\int \mathrm{d}\Phi_b\, \mathcal{K}(\Phi_a, \Phi_b)\mathrm{e}^{\frac{\mathrm{i}}{8}g(\Phi_b)+f(\Phi_b)}}{\int \mathrm{d}\Phi_b\, \mathrm{e}^{\frac{\mathrm{i}}{8}g(\Phi_b)+f(\Phi_b)}}. \tag{49}$$

When deriving equation (49) we have used (45). It is completely clear that in the limit $N \to \infty$ one can safely disregard the term $f(\Phi_b)$, equation (42) as being of the next order in $1/N$ in comparison with $g(\Phi_b)$. From now on we just put $f = 0$ everywhere.

Given the form of the kernel equation (41), we managed to guess the following explicit solution to the saddle-point equation:

$$g_s(\Phi_a) = 4(r - G_1)(\Phi_a^\dagger \hat{\Lambda} \Phi_a) + 4\mathrm{i}G_2(\Phi_a^\dagger \Phi_a) + \mathrm{i}(\Phi_a^\dagger \hat{\Lambda} \Phi_a)^2. \tag{50}$$

Substituting such an ansatz into equation (49) by a direct calculation one can verify that it indeed satisfies the saddle-point equation provided the real coefficients G_1, G_2 are solutions of the system of two conjugate equations:

$$\begin{aligned} G_2 + \mathrm{i}G_1 &= \int_0^\infty \mathrm{d}u \exp\left\{\frac{\mathrm{i}}{2}u(r - G_1 + \mathrm{i}G_2) - \frac{u^2}{8}\right\} \\ G_2 - \mathrm{i}G_1 &= \int_0^\infty \mathrm{d}u \exp\left\{-\frac{\mathrm{i}}{2}u(r - G_1 - \mathrm{i}G_2) - \frac{u^2}{8}\right\}. \end{aligned} \tag{51}$$

The following few identities might be helpful when performing a verification of such a fact. Suppose we have a four-component supervector Ψ like that defined in equation (20) and consider a function $F(\Psi) = \tilde{F}(\Psi^\dagger \Psi)$ vanishing on the boundary of integration. Then it is easy to verify that

$$\int \mathrm{d}\Psi\, F(\Psi) = \tilde{F}(0) \qquad \int \mathrm{d}\Psi_b\, (\Psi_a^\dagger \Psi_b) F(\Psi_b) = 0 \tag{52}$$

where the first identity is just a particular case of the PSEW theorem mentioned in the previous section: see, e.g. [36].

For further analysis it is very important that a solution for system (51) exists for arbitrary $-\infty < r < \infty$ such that $G_2(r) > 0$. Later on we see that the mean density of resonances is merely given by $\rho(r) = \frac{1}{\pi}G_2(r)$. For $|r| \gg 1$, one can easily infer from equations (51) that $G_2(r)$ developes a Gaussian tail:

$$G_2(r \gg 1) \approx (\pi/2)^{1/2} \exp\left\{-\frac{r^2}{2}\right\}.$$

It is necessary to mention that equations (51) virtually coincide with those emerging in a study of density of eigenvalues of a transition matrix on a randomly diluted graph in a limit of large connectivity performed by Bray and Rodgers [38]: see their equations (20), (21) and figure 1 in their paper. Indeed, two problems have many common features, and the coincidence is hardly accidental. An indication of a relation between the problems comes from the structure of the matrix $\hat{\boldsymbol{H}}$, see equation (9), which is actually very similar to the structure of the matrix considered by Bray and Rodgers. Moreover, the resonances we study are eigenvalues of the matrix $\tilde{\boldsymbol{H}} = \boldsymbol{W}^{-1/2}\hat{\boldsymbol{H}}\boldsymbol{W}^{-1/2}$, and it is easy to write down explicit expressions for the entries of $\tilde{\boldsymbol{H}}$ because of the simple structure of the matrix $\boldsymbol{W}$, see equation (9). One finds that $\tilde{\boldsymbol{H}} = \frac{1}{N}\boldsymbol{H} + \mathrm{O}(1/N^{3/2})$ which provides a direct link to [38] in the limit $N \to \infty$.

The most important consequence of the existence of such a solution $g_s(\Phi_a)$, see equation (50), with $G_2 \neq 0$ is actually the simultaneous existence of a whole *continuous manifold* of the saddle-point solutions parametrized as

$$g_T(\Phi_a) = g_s(\hat{T}\Phi_a) \qquad \text{with} \quad \hat{T}^\dagger \hat{\Lambda} \hat{T} = \hat{\Lambda}. \tag{53}$$

Indeed, from the invariance property of the kernel equation (41): $\mathcal{K}(\Phi_a, \Phi_b) = \mathcal{K}(\hat{T}\Phi_a, \hat{T}\Phi_b)$ and from the (pseudo)unitarity of the matrices T: $|\mathrm{Sdet}\hat{T}| = 1$ it follows that $g_s(\hat{T}\Phi_a)$ must be a solution to equation (49) together with any given solution $g_s(\Phi_a)$. However, if it had not been for the condition $G_2 \neq 0$ all these solutions would trivially coincide: $g_T(\Phi) \equiv g(\Phi)$ for any $\hat{T}$ defined as above, i.e. the symmetry of the solution would coincide with the symmetry of the equation itself.

In the actual case $G_2 \neq 0$, the presence of the combination $\Phi_a^\dagger \Phi_a$ which is *not invariant* with respect to a transformation $\Phi_a \to \hat{T}\Phi_a$ makes the symmetry of solution (50) lower than the symmetry of equation (49). This is an example of the very well known effect of spontaneous symmetry breakdown. The phenomenon makes the situation less trivial and generates the whole manifold of the saddle-point solutions. Different nontrivial solutions are actually parametrized by the supermatrices $\hat{T}$ which are elements of the same† graded coset space $UOSP(2,2/4)/UOSP(2/2) \otimes UOSP(2/2)$ which appeared in the previous section: see the discussion after equation (23).

It is simple to satisfy oneself that $\mathcal{L}(g_s) = 0$, and the same obviously holds for the whole manifold g_T. Therefore, the only term that renders the expression for the generation function to be non-trivial is $\delta\mathcal{L}_1(g_T)$. In the limit $(\epsilon, J, 1/N) \to 0$ we can expand equation (48) as

$$\delta\mathcal{L}_1(g_T) = \ln\left\{1 + \frac{\mathrm{i}}{2N^{1/2}} \sum_{i<j} \int \mathrm{d}\Phi_i\, \mathrm{d}\Phi_j\, (\Phi_i - \Phi_j)^\dagger (\mathrm{i}\tilde{\epsilon}\hat{1} + \hat{J})(\Phi_i - \Phi_j) \mathrm{e}^{\frac{\mathrm{i}}{8}(g_T(\Phi_i) + g_T(\Phi_j))}\right\}$$

$$\approx \frac{\mathrm{i}}{2} N^{3/2} \int \mathrm{d}\Phi\, \Phi^\dagger (\mathrm{i}\tilde{\epsilon}\hat{1} + \hat{J}) \Phi \mathrm{e}^{\frac{\mathrm{i}}{8} g_T(\Phi)} = \frac{\mathrm{i}}{2} N^{3/2}\, \mathrm{Str}[\hat{W}(\mathrm{i}\tilde{\epsilon}\hat{1} + \hat{J})] \tag{54}$$

where we introduced a supermatrix:

$$W_{\alpha\beta} = \int \mathrm{d}\Phi\, \Phi_\alpha \Phi_\beta^\dagger \mathrm{e}^{\frac{\mathrm{i}}{8} g_T(\Phi)} \tag{55}$$

and used the fact that for our purposes it is enough to keep only terms linear with respect to the source matrix $\hat{J}$ (see the discussion after equation (27)).

To determine the matrix elements of $\hat{W}$ it is convenient to use the equation

$$G_2 \Phi_a^\dagger \Phi_a + \mathrm{i} G_1 \Phi_a^\dagger \hat{\Lambda} \Phi_a = \int \mathrm{d}\Phi_b (\Phi_a^\dagger \hat{\Lambda} \Phi_b)(\Phi_b^\dagger \hat{\Lambda} \Phi_a) \mathrm{e}^{\frac{\mathrm{i}}{8} g_s(\Phi_b)} \tag{56}$$

which follows from the saddle-point equation (49) and equation (50). Now we make a transformation: $\Phi_{a,b} \to \hat{T}\Phi_{a,b}$, and easily find that

$$G_2 \Phi_a^\dagger \hat{T}^\dagger \hat{T} \Phi_a + \mathrm{i} G_1 \Phi_a^\dagger \hat{\Lambda} \Phi_a = \int \mathrm{d}\Phi_b\, (\Phi_a^\dagger \hat{\Lambda} \Phi_b)(\Phi_b^\dagger \hat{\Lambda} \Phi_a) \mathrm{e}^{\frac{\mathrm{i}}{8} g_T(\Phi_b)}$$

† The following comment is appropriate here. Actually, the condition $\hat{T}^\dagger \hat{\Lambda} \hat{T} = \hat{\Lambda}$ defines a graded group $UOSP(2,2/2,2)$ which is slightly different from $UOSP(2,2/4)$, since the latter requires a *compact* parametrization in the 'fermionic' sector, whereas the former is 'non-compact' in both 'bosonic' and 'fermionic' sectors. However, a detailed investigation of the structure of the saddle-point manifold and the necessity to perform an integration over matrices $\hat{T}$ forces one to choose these matrices to be 'compactified' in the 'fermionic' sector. This is a subtle feature discussed in much detail in [35]. In principle, such an *ad hoc* compactification can be avoided if one uses two different matrices: $\hat{\Lambda}$ and $\hat{L} = \mathrm{diag}(1,1,1,1,-1,-1,1,1)$ when writing an integral representation for the generating function, equation (16). Such a freedom of choice always exists because Grassmann integrals are always convergent. Implementation of such a program can be found, e.g. in [26]. Practically, however, the results are the same as if one 'compactified' the matrices $\hat{T}$ when performing the actual calculations.

$$= \sum_{\alpha,\beta=1}^{8} \Phi_{a,\alpha}\Phi^{\dagger}_{a,\beta}\Lambda_{\alpha\alpha}\Lambda_{\beta\beta}K_{\alpha\alpha}W_{\beta\alpha} \tag{57}$$

where $K_{\alpha\alpha} = -1$ for α being a 'fermionic' index and unity otherwise. Comparing both sides of this relation (i.e. the coefficients of the quadratic forms with respect to the components of the supervector Φ_a) yields a relation between W, the functions $G_{1,2}$ and the matrices $\hat{T}$:

$$\hat{W} = G_2\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda} + \mathrm{i}G_1\hat{\Lambda} \tag{58}$$

where we used $\hat{T}^{\dagger}\hat{T} = \Lambda\hat{T}^{-1}\hat{\Lambda}\hat{T}$.

Using this fact we find that the averaged generating function is expressed as an integral over the saddle-point manifold parametrized by the matrices $\hat{T}$:

$$\langle \mathcal{Z}^{r,s}(J_1, J_2)\rangle = \int \mathrm{d}\mu\,(T) \times \exp\left\{\mathrm{i}\frac{s}{4}G_2(r)\,\mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}) + \frac{\mathrm{i}}{2}N^{3/2}\,\mathrm{Str}\,\hat{J}_k(G_2\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda} + \mathrm{i}G_1\hat{\Lambda})\right\} \tag{59}$$

with $\mathrm{d}\mu\,(T)$ being the invariant (Haar's) measure on the graded coset space. Here we assumed an infinitesemal negative imaginary part to be included in s. Remembering that $\hat{J}_k = J_1\hat{k}\frac{1+\hat{\Lambda}}{2} + J_2\hat{k}\frac{1-\hat{\Lambda}}{2}$ and making the correspondence

$$G_1(r) \longrightarrow \frac{1}{N}\sum_i \frac{\lambda - v_i t}{(\lambda - v_i t)^2 + q^2 v_i^2} \qquad \text{and} \qquad G_2(r) \longrightarrow \frac{1}{N}\sum_i \frac{q v_i}{(\lambda - v_i t)^2 + q^2 v_i^2} \tag{60}$$

we see that the expression obtained literally coincides† with the averaged generation function derived in the previous section, that means with the generating function equation (24) after restricting the integration to the saddle-point manifold equation (32) and expanding with respect to source terms. Since that generating function underlied the expressions (37) for the two-point spectral correlation function as well as expression (30) for the mean spectral density, we immediately extract those quantities for our actual problem. As already mentioned above, the mean resonance density is given by $\rho(r) = \frac{1}{\pi}G_2(r)$, and the functional form of the spectral correlation function is given by the same Wigner–Dyson expression as before, see equation (38), provided one uses $\rho(r)$ for the actual mean resonance density.

Finally, it is necessary to mention that by using the same method it is straightforward to demonstrate that all higher spectral correlation functions will also be given by GOE expressions.

4. Conclusion

In the present paper we introduced the full-connectivity model of a disordered reactance LC network and found that its spectral properties in the thermodynamic limit $N \to \infty$ could be efficiently investigated in the framework of a version of Efetov's supersymmetry method [34–36] exploiting a generalized Hubbard–Stratonovich transformation introduced by us earlier [22]. We also studied spectral properties of regular pencils of random matrices with IID entries. In all cases we were able to derive the mean spectral density as well as characterize the fluctuation properties of the spectra. The models studied turned out to be faithful representatives of the Wigner–Dyson universality class.

† An extra factor $N^{1/2}$ appearing in the exponent equation (59) as compared with similar equations of the previous paragraph is due to our rescaling: $\tilde{\lambda} \to r/N^{1/2}$ which also requires the spectral density to be redefined appropriately: $\rho(r) = N^{-1/2}\rho(\tilde{\lambda})$.

Actually, we hope that the present paper may provide a convenient background for the regular investigation of fluctuations in the electric properties of disordered LC (more generally, RL–C) networks, such as the conductance Y_{AB} defined in equation (8), on-site potentials V_i, etc. Actually, it is possible to consider an LC-model of a 'banded' type (i.e. of a large, but finite range: see [39,40]) and derive [42] the corresponding d-dimensional nonlinear σ-model which should adequately describe the properties of the realistic LC-networks [5,7], including the effects of Anderson localization [41]. These issues will be addressed in forthcoming publications [42].

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Appendix. A digression on functional Hubbard-Stratonovich transform, equation (44)

In this appendix we present a heuristic demonstration of the validity of relation (44) for some kernels $\mathcal{K}$. We are not pretending that we are able to specify under what precise conditions our formal manipulations are true: this interesting issue deserves to be studied seriously and goes much beyond our modest goals. Instead, our strategy will be a pragmatic one: to illuminate the origin of the relations of that kind.

Actually, the fact that Φ is a supervector rather than a usual vector is of little importance for our consideration, so one may think about it as a usual vector (or even scalar). The way to incorporate anticommuting variables in the consideration sketched below is described in the appendix A of [39].

Let us then deal with a real symmetric integral kernel $\mathcal{K}(\Phi_a, \Phi_b) = \mathcal{K}(\Phi_b, \Phi_a)$. Then, generally, we expect a set of real eigenvalues λ_ν to exist and an orthogonal and normalizable set of corresponding real-valued eigenfunctions $e_\nu(\Phi)$:

$$\int \mathrm{d}\Phi_b \, \mathcal{K}(\Phi_a, \Phi_b) e_\nu(\Phi_b) = \lambda_\nu e_\nu(\Phi_a) \qquad \int \mathrm{d}\Phi \, e_\nu(\Phi) e_\mu(\Phi) = \delta_{\mu\nu} \quad \text{(A1)}$$

such that the kernel $\mathcal{K}$ allows the following representation:

$$\mathcal{K}(\Phi_a, \Phi_b) = \sum_\nu \lambda_\nu e_\nu(\Phi_a) e_\nu(\Phi_b) \quad \text{(A2)}$$

where the summation goes over all ν such that $\lambda_\nu \neq 0$. Then we can write a formal chain of transformations:

$$I = \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{N} \mathcal{K}(\Phi_i, \Phi_j) \right\} = \prod_{i,j} \exp\left\{ -\frac{1}{2} \sum_\nu \lambda_\nu e_\nu(\Phi_i) e_\nu(\Phi_j) \right\}$$

$$= \prod_\nu \exp\left\{ -\frac{\lambda_\nu}{2} \left(\sum_{i=1}^{N} e_\nu(\Phi_i) \right)^2 \right\}$$

$$= \prod_{\nu} \int_{-\infty}^{\infty} \frac{\mathrm{d}z_\nu}{\sqrt{2\pi}} \exp\left\{ -\frac{z_\nu^2}{2} - \mathrm{i}\lambda_\nu^{1/2} z_\nu \sum_{i=1}^{N} e_\nu(\Phi_i) \right\} \tag{A3}$$

$$= \int \cdots \int \left(\prod_{\nu} \frac{\mathrm{d}z_\nu}{\sqrt{2\pi}} \right) \exp\left[-\frac{1}{2} \sum_{\nu} z_\nu^2 - \mathrm{i} \sum_{i=1}^{N} \left(\sum_{\nu} \lambda_\nu^{1/2} z_\nu e_\nu(\Phi_i) \right) \right]. \tag{A4}$$

Again, we do not specify the conditions when it is allowed to interchange the summations, products and integrations but just assume 'simple-mindedly' that the sequence of transformations presented above does not lead to divergent expressions.

A form of expression (A4) suggests that one should consider it as a definition of a functional integral going over the space of functions

$$g(\Phi) = \sum_{\nu} \lambda_\nu^{1/2} z_\nu e_\nu(\Phi) \qquad \mathcal{D}(g) \equiv \prod_{\nu} \frac{\mathrm{d}z_\nu}{\sqrt{2\pi}}. \tag{A5}$$

Finally, we introduce the kernel $\mathcal{C}(\Phi_a, \Phi_b)$ as

$$\mathcal{C}(\Phi_a, \Phi_b) = \sum_{\mu} \lambda_\mu^{-1} e_\mu(\Phi_a) e_\mu(\Phi_b) \tag{A6}$$

which obviously satisfies

$$\int \mathrm{d}\Phi\, \mathcal{C}(\Phi_a, \Phi) \mathcal{K}(\Phi, \Phi_b) = \sum_{\mu} e_\mu(\Phi_a) e_\mu(\Phi_b) \equiv \delta_g(\Phi_a, \Phi_b) \tag{A7}$$

where the $\delta_g(\Phi_a, \Phi_b)$ plays the role of a δ-function in the space spanned by functions $g(\Phi)$, equation (A5):

$$\int \mathrm{d}\Phi_a\, g(\Phi_a) \delta_g(\Phi_a, \Phi_b) = g(\Phi_b).$$

Moreover, using equation (A5) it is straightforward to verify that

$$\int \mathrm{d}\Phi_b\, \mathrm{d}\Phi_a\, g(\Phi_a) \mathcal{C}(\Phi_a, \Phi_b) g(\Phi_b) = z_\nu^2$$

which finally allows us to rewrite equations (A3), (A4) as

$$\exp\left\{ -\tfrac{1}{2} \sum_{i,j=1}^{N} \mathcal{K}(\Phi_i, \Phi_j) \right\} = \int \mathcal{D}(g) \times \exp\left\{ -\tfrac{1}{2} \int \mathrm{d}\Phi_b\, \mathrm{d}\Phi_a\, g(\Phi_a) \mathcal{C}(\Phi_a, \Phi_b) g(\Phi_b) - \mathrm{i} \sum_{i=1}^{N} g(\Phi_i) \right\} \tag{A8}$$

which can be brought to the form of equation (44) by a trivial scaling transformation.

Let us finally note, that our actual kernel equation (41) is of a separable nature and, moreover, has only a few nonzero eigenvalues. Evidently, in that case the formal manipulations, equation (A3), are expected to be most harmless.

References

[1] Dykhne A M 1970 *Zh. Eksp. Teor. Fiz.* **59** 110 (Engl. transl. 1970 *Sov.Phys.–JETP* **59** 110)
[2] Clerk J P, Giraud G, Laugier J M and Luck J M 1990 *Adv. Phys.* **39** 191
[3] Shender E F 1978 *J. Phys. C: Solid State Phys.* **11** L423
[4] Kirkpatrick S 1973 *Rev. Mod. Phys.* **45**
[5] Brouers F, Blacher S, Lagarkov A N, Sarychev A K, Gadenne P and Shalaev V M 1997 *Phys. Rev.* B **55** 13 234
[6] Clerk J P, Giraud G, Luck J M and Robin Th 1996 *J. Phys. A: Math. Gen.* **29** 4781
[7] Jonckheere Th and Luck J M 1998 *J. Phys. A: Math. Gen.* **31** 3687

[8] Straley J P 1979 *J. Phys. C: Solid State Phys.* **12** 2143
[9] Luck J M 1991 *Phys. Rev.* B **43** 3933
[10] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **32** 1792
[11] Gantmacher F 1959 *Theory of Matrices* vol 2 (New York: Chelsea)
[12] Kagström B and Ruhe A 1983 *Matrix Pencils (Lecture Notes in Mathematics 973)* (New York: Springer)
[13] Thompson G L and Well R L 1972 *Lin. Alg. Appl* **5** 207
[14] Edelman A, Kostlan E and Shub M 1994 *J. Am. Math. Soc.* **7** 247
[15] Marchenko V A and Pastur L A 1967 *Math. USSR-Sbornik* **1** 457
Silverstein J 1985 *SIAM J. Math. Anal.* **16** 641
Bai Z, Yin Y and Krishnaiah P 1987 *Theor. Prob. Appl.* **32** 490
Silverstein J and Bai Z 1995 *J. Multiv. Anal.* **54** 175
Silverstein J 1995 *J. Multiv. Anal.* **55** 331
[16] Pastur L A 1972 *Theor. Math. Phys.* **10** 67
See also Pastur L 1996 *Ann. Inst. H. Poincaré A* **64** 325
Girko V 1995 *Theor. Prob. Appl.* **39** 685
[17] Guhr T, Müller-Groelling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 3439
[18] Pastur L and Shcherbina M 1997 *J. Stat. Phys.* **86** 109
[19] Ambjorn J, Jurkiewicz J and Makeenko Y 1990 *Phys. Lett.* B **251** 517
Brezin E and Zee A 1993 *Nucl. Phys.* B **402** 613
[20] Hackenbroich G and Weidenmüller H A 1995 *Phys. Rev. Lett.* **74** 4118
[21] Brezin E and Hikami S 1998 *Phys. Rev.* E **58** 7176
[22] Mirlin A D and Fyodorov Y V 1991 *J. Phys. A: Math. Gen.* **24** 2273
Fyodorov Y V and Mirlin A D 1991 *Phys. Rev. Lett.* **67** 2049
Fyodorov Y V and Mirlin A D 1991 *J. Phys. A: Math. Gen.* **24** 2219
[23] Khorunzhy A M, Khoruzhenko B A and Pastur L A *J. Math. Phys.* **37** 5033
[24] Pastur L and Figotin A 1992 *Spectra of Random and Almost-Periodic Operators* appendix B (Berlin: Springer)
[25] Rodgers G J and Bray A J 1988 *Phys. Rev.* B, **37** 3557
[26] Fyodorov Y V and Sommers H-J 1995 *Z. Phys.* B **99** 123
[27] Akemann G 1997 *Nucl. Phys.* B **507** 475
[28] Mehta M L 1990 *Random Matrices* (New York: Academic)
[29] Evangelou S 1992 *J. Stat. Phys.* **69** 361
[30] Bohigas O 1991 *Chaos and Quantum Physics (Proc. Les Houches Summer School Session LII)* ed M J Giannoni *et al* (Amsterdam: North-Holland) p 91
[31] Altshuler B L and Simon B D 1995 *Mesoscopic Quantum Physics (Les Houches Summer School, Session LXI)* ed E Akkermans *et al* (Amsterdam: North-Holland)
[32] Berry M 1985 *Proc. R. Soc.* A **400** 229
Bogomolny E and Keating J 1996 *Phys. Rev. Lett.* **77** 1472
[33] Muzykantsky B A and Khmelnisky D E 1995 *JETP Lett.* **62** 76
Andreev A, Agam O, Altshuler B and Simons B 1996 *Phys. Rev. Lett.* **76** 3947
[34] Efetov K 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[35] Verbaarschot J J M, Weidenmüller H A and Zirnbauer M R 1985 *Phys. Rep.* **129** 367
[36] Fyodorov Y V 1995 *Mesosocopic Quantum Physics (Les Houches Summer School, Session LXI, 1994)* ed E Akkermans *et al* (Amsterdam: Elsever) p 493
[37] Altland A, Efetov K B and Iida S 1993 *J. Phys. A: Math. Gen.* **26** 2545
[38] Bray A J and Rodgers G J 1988 *Phys. Rev.* B **38** 11 461
[39] Fyodorov Y V, Mirlin A D and Sommers H-J 1992 *J. Physique* I **2** 1571
[40] Fyodorov Y V and Mirlin A D 1991 *Phys. Rev. Lett.* **67** 2405
[41] Gresillon S *et al* 1999 *Phys. Rev. Lett.* **82** 4520
Sarychev A K and Shalaev V M 1999 *Physica* A **266** 115
[42] Fyodorov Y V 1999 Fluctuations in random RL–C networks *Preprint*